$\therefore$ 190.90 $> x+(76800-8x^2)/5x$. $\therefore$ 190.90 $>(76800-3x^2)/5x$.
As $x$ decreases, $(76800-3x^2)/5x$ increases.

$\therefore$ the equation $(76800-3x^2)/5x=190.90$ ..........................(4)

gives the minimum limit of $x$.

$\therefore$ 66.48+ is the minimum limit of $x$......... ....................(5).

From (3), $y=(76800-8x^2)/5x$, we get, since $y$ must have some value, $76800>8x^2$; hence $8x^2=76800$ gives maximum limit of $x$. $\therefore$ 97.97+ is the maximum limit of $x$. Hence, any values of $x$ between 66.48+ and 97.97+ will satisfy the conditions of the problem. *Example:* Let $x=77\frac{1}{7}$. Then from (3) $y=75\frac{43}{63}$; $\therefore z=47\frac{11}{63}$.

$\therefore$ A received \$136.65$\frac{135}{441}$; B received \$112.17$\frac{103}{441}$. $\therefore$ C received \$38.07$\frac{203}{441}$; but he paid \$47.17$\frac{203}{441}$. $\therefore$ C lost \$9.10.

Also solved by *A. H. HOLMES*, *J. SCHEFFER*, and *G. B. M. ZERR*.

## PROBLEMS.

**72. Proposed by CHAS. C. CROSS, Laytonsville, Maryland.**

Prove that $\dfrac{2\sqrt{2+\sqrt{3}}}{4\times\sqrt{6-\sqrt{2}}}=\sqrt{6}-\sqrt{2}+\sqrt{3}-2$, when reduced to its lowest terms.

**73. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.**

Find the worth of each of five persons, A, B, C, D, and E, knowing, 1st, that when A's worth is added to $a$ times what B, C, D, and E are worth, it is equal to $m$; 2nd, when B's worth is added to $b$ times what A, C, D, and E are worth, it is equal to $n$; 3rd, when C's worth is added to $c$ times what A, B, D, and E are worth, it is equal to $p$; 4th, when D's worth is added to $d$ times what A, B, C, and E are worth, it is equal to $q$; 5th, when E's worth is added to $e$ times what A, B, C, and D are worth, it is equal to $r$.

# CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

## SOLUTIONS OF PROBLEMS.

**51. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.**

Find the maximum ellipsoid that can be cut out of a given right conic frustum.

I. **Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.**

A complete solution of this problem without any assumptions would be a task greater than I care to undertake at present. We will, therefore, assume the cone to be one of revolution. Let $2h$=height of frustum, $R$, $r$ radii of the lower and upper bases, respectively, $l$, $m$, $p$ the coordinates of the vertex.

$x^2/a^2+y^2/b^2+z^2/c^2=1$, the equation to the ellipsoid.

$\therefore$ $(p-z)^2+(n-y)^2=[(R-r)/2h]^2(m-x)^2$ is the equation to the cone.

We will further assume that this cone is the tangent cone to the maximum ellipsoid, then the equation to the cone is

$$(m^2/a^2+n^2/b^2+p^2/c^2-1)(x^2/a^2+y^2/b^2+z^2/c^2-1)$$
$$=(mx/a^2+ny/b^2+pz/c^2-1)^2.$$

From these two equations to the cone we get $n=p=0$.

$[(R-r)/2h]^2=Rr/(m^2-h^2)$ or $m=[(R+r)/(R-r)]h$.

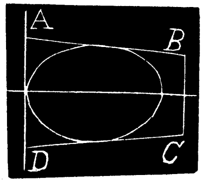
$\therefore$ The center of the frustum and the center of the ellipsoid coincide, and the ellipsoid is one of revolution.

$\therefore$ $x^2/a^2+(y^2+z^2)/b^2=1$ is its equation. $\therefore$ $a=h$, $b=\sqrt{Rr}$.

$V=\frac{4}{3}\pi hRr$=volume of maximum ellipsoid.

II. **Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.**

The figure shows vertical section of frustum and inscribed ellipsoid, with axis of $x$ coinciding with axis of cone, and axis of $y$ in base. Let $d$ and $c$ be radii of bases, and $h$ the altitude. Then $(0, d)$ and $(h, c)$ represent points $A$ and $B$, respectively.




$(x-a)^2/a^2+y^2/b^2=1$ is equation to inscribed ellipse. $\therefore$ equation to $AB$, as tangent to ellipse, is

$$a^2yy_1+b^2(x-a)(x_1-a)=a^2b^2 \quad\ldots\ldots\ldots(1),$$

$x_1$ and $y_1$ being coordinates of point of contact. Substituting coordinates of $A$ and $B$ for $x$ and $y$ in (1),

$$\left.\begin{array}{l} a^2dy_1+b^2(-x)(x_1-a)=a^2b^2 \\ a^2cy_1+b^2(h-a)(x_1-a)=a^2b^2 \end{array}\right\} \quad\ldots\ldots\ldots(2).$$

Solving (2) for $x_1$ and $y_1$,

$$\left.\begin{array}{l} x_1=ahd/(bd-ad+ac) \\ y_1=b^2h/(bd-ad+ac) \end{array}\right\} \quad\ldots\ldots\ldots(3).$$

Substituting from (3) for $x$ and $y$ in equation of ellipse and solving we obtain $b^2=[(hd^2+2ad(c-d)]/h$.

Now volume of ellipsoid $V=4/3(\pi ab^2)=4\pi/3h[ad^2h+2a^2d(c-d)]$.

$$dV/da=4\pi/3h[d^2h+4ad(c-d)]\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Equating (4) to 0, we find $a=dh/4(d-c)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$,

$$\text{and } b^2=d^2/2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Also, $d^2V/da^2=16\pi d(c-d)/3h$, which is negative since $d>c$. Now the ellipsoid will be entirely within the frustum if $2a$ is not greater than $h$, which from (5) gives, $dh/2(d-c)$ is not greater than $h$ or $c$ is not greater than $\frac{1}{2}d$. So volume of maximum ellipsoid=

$\frac{4\pi}{3}\cdot\frac{dh}{4(d-c)}\cdot\frac{d^2}{2}=\frac{\pi}{6}\frac{d^3h}{d-c}$, if $c$ is not greater than $\frac{1}{2}d$, or $\frac{4\pi}{3}hb^2$, if $c>\frac{1}{2}d$, the latter result being true, since (4) shows but one maximum, and $V$ is a continuous function of $A$.

**III. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.**

Take the base of the frustum as the plane $xz$, and the axis of the frustum as the axis of $y$. We may, without loss of generality, take one axis parallel to the axis of $z$. The equation of the ellipsoid may then be written:

$$Ax^2+By^2+Cxy+Dx+Ey+Hz^2+F=0\ldots\ldots\ldots\ldots\ldots(1).$$

We find the axes of the ellipsoid to be:

$$a=\sqrt{R/P},\ b=\sqrt{R/Q},\ c=\sqrt{R/H},$$

where $R=F(c^2-4AB)+AE^2+BD^2-CD^2)/(4AB-C^2)$.

$P=1/2[A+B\pm\sqrt{(A-B)^2+C^2}]$,

$Q=1/2[A+B\mp\sqrt{(A-B)^2+C^2}]$,

Volume of ellipsoid$=4/3(\pi abc)$

$$=\tfrac{8}{3}\pi\frac{[F(C^2-4AB)+AE^2+BD^2-CDE]^{\frac{3}{2}}}{[4AB-C^2]^2}\cdot\frac{1}{\sqrt{H}}\ldots\ldots\ldots\ldots(2).$$

A little consideration will show that the ellipsoid to be a maximum *must touch* the larger base of the frustum and also the conical surface. The condition that it touch the lower base is $D^2-4AF=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$.

The condition that it shall not cut the upper base is

$$(Ch+D)^2-4A(Bh^2+Eh+F(<0\ \ldots\ldots\ldots\ldots\ldots(4),$$

where $h$ is the altitude of the frustum.

To find the condition that the ellipsoid shall be tangent to conical surface, we assume the equation of complete cone to be:

$$m^2(x^2+z^2)=(y-k)^2 \ldots\ldots \ldots\ldots\ldots\ldots \ldots\ldots (5).$$

For intersection of (1) and (5),

$$(A-H)m^2x^2+(Bm^2+H)y^2+Cm^2xy+Dm^2x+(Em^2-2k)+(Hk^2+Fm^2)=0.$$

If this ellipse have no axes,

$$(Hk^2+Fm^2)[c^2m^2-4(A-H)(Bm^2+H)]+(A-H)(Em^2-2k)^2+(Bm^2+H)D^2m^2$$
$$-CD(Em^2-2k)m^2=0.$$

Solving this for $B$ we obtain,

$$B=\frac{CD(Em^2-2k)m^2-(A-H)(Em^2-2k)^2-C^2m^2(Hk^2+Fm^2)}{m^2[D^2m^2-4(A-H)(Hk^2+Fm^2)]}-\frac{H}{m^2}.$$

Substitute the value of $A$ given in (3),

$$B=\frac{4FCD(Em^2-2k)m^2-(D^2-4FH)(Em^2-2k)^2-4FC^2m^2(Hk^2+Fm^2)}{m^2[FD^2m^2-(D^2-4FH)(Hk^2+Fm^2)]}-\frac{H}{m^2}.$$

If we were then to substitute these values of $A$ and $B$ in equation (2), we should obtain a value of $V$ which contains the variables $C$, $D$, $E$, $F$, and $H$, independent, except as to the condition given in (4). By the ordinary methods of maximum and minimum, five equations can be formed and the maximum critical values of the five letters determined. But life is too short to do this.

If we assume that two of the axes are parallel to the bases of the frustum, we obtain $V=\frac{4}{3}\pi\tan^2\phi(h^2b-2hb^2)$, where $h$=altitude of complete cone, $\phi$=semi-angle of cone, and $b$=semi-vertical axis of ellipsoid. From this for maximum, $b=p/4$.

**52. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.**

There are two lights of intensities $m$ and $n$. Where must a target, whose surface is parallel to the line joining the two lights, be set up in order that it shall receive the maximum illumination per unit of area?

**I. Solution by the PROPOSER.**

If we take the point where the light with intensity $l$ is situated as the origin of coordinates, we have readily from the principles of Optics,
$I=ly/(x^2+y^2)^{\frac{3}{2}}+my/[(a-x)^2+y^2]^{\frac{3}{2}}$, $x$ and $y$ being the coordinates of the bull's-eye.